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TOPOLOGICAL REALIZATIONS OF FAMILIES OF ERGODIC AUTOMORPHISMS, MULTITOWERS AND ORBIT EQUIVALENCE

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ABSTRACT

We study minimal topological realizations of families of ergodic measure preserving automorphisms (e.m.p.a.'s). Our main result is the following theorem.

THEOREM: *Let $\{T_p : p \in \mathcal{I}\}$ be an arbitrary finite or countable collection of e.m.p.a.'s on nonatomic Lebesgue probability spaces (Y_p, ν_p). Let S be a Cantor minimal system such that the cardinality of the set $\mathcal{E}_S$ of all ergodic S-invariant Borel probability measures is at least the cardinality of $\mathcal{I}$. Then for any collection $\{\mu_p : p \in \mathcal{I}\}$ of distinct measures from $\mathcal{E}_S$ there is a Cantor minimal system S' in the topological orbit equivalence class of S such that, as a measure preserving system, (S', μ_p) is isomorphic to T_p for every $p \in \mathcal{I}$. Moreover, S' can be chosen strongly orbit equivalent to S if and only if all finite topological factors of S are measure-theoretic factors of T_p for all $p \in \mathcal{I}$.*

This result shows, in particular, that there are no restrictions at all for the topological realizations of countable families of e.m.p.a.'s in Cantor minimal systems. Namely, for any finite or countable collection $\{T_1, T_2, \ldots\}$ of e.m.p.a.'s of nonatomic Lebesgue probability spaces, there is a Cantor

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minimal system S, whose collection $\{\mu_1, \mu_2, \ldots\}$ of ergodic Borel probability measures is in one-to-one correspondence with $\{T_1, T_2, \ldots\}$, and such that (S, μ_i) is isomorphic to T_i for all i.

Furthermore, since realizations are taking place within orbit equivalence classes of a given Cantor minimal system, our results generalize the strong orbit realization theorem and the orbit realization theorem of [18]. Those theorems are now special cases of our result where the collections $\{T_p\}$, $\{\mu_p\}$ consist of just one element each.

1. Introduction

The problem of constructing topological models for ergodic measure preserving transformations has been studied for quite a long time. For a single transformation T of a Lebesgue probability space the natural class of topological systems in which one should try to find a model for T is the class of uniquely ergodic homeomorphisms of compact metric spaces. The famous Jewett–Krieger theorem ([14], [15]) states that an *arbitrary* ergodic automorphism T of a nonatomic Lebesgue space has a topological model in the class of Cantor minimal and uniquely ergodic systems (strictly ergodic homeomorphisms of a Cantor set).

It is well known that minimal systems need not be uniquely ergodic and may have many invariant measures. In fact, as was shown by T. Downarowicz [3], for arbitrary Choquet metrizable simplex Δ there is a Cantor minimal system S whose simplex of all Borel probability invariant measures (in the weak*-topology) is affinely homeomorphic to Δ. Given a Cantor minimal system S with the collection $\{\mu_\alpha : \alpha \in \mathcal{I}\}$ of its ergodic invariant measures, one naturally associates to it the family $\{T_\alpha : \alpha \in \mathcal{I}\}$ of e.m.p.a.'s, where $T_\alpha = (S, \mu_\alpha)$. A question then arises of characterization of those families $\{T_\alpha\}$ of e.m.p.a.'s that can be obtained, up to isomorphism, in this way. More precisely, given a family $\{T_\alpha : \alpha \in \mathcal{I}\}$ of e.m.p.a.'s, one wants to decide whether or not there is a Cantor minimal system S with the collection of its ergodic invariant measures $\{\mu_\alpha : \alpha \in \mathcal{I}\}$ in 1-to-1 correspondence with $\{T_\alpha\}$ such that measure-theoretically the system (S, μ_α) is isomorphic to T_α for every $\alpha \in \mathcal{I}$.

In recent years, a number of results related to this problem have been obtained. B. Weiss [21] gave an example of a *universal minimal system*, i.e., a minimal homeomorphism S: $X \to X$ of a Cantor set X such that given any aperiodic automorphism T of a Lebesgue probability space (Y, ν), there exists a S-invariant Borel probability measure μ such that (X, S, μ) and (Y, T, ν) are measurably isomorphic. We also mention the result of T. Downarowicz and

J. Serafin [7], which characterized the class of functions that can occur as the entropy functions defined on the space of all Borel probability invariant measures of a topological (Cantor) dynamical system. In the papers of Downarowicz and Lacroix [6] and Downarowicz and Durand [5] the results on minimal realizations of families of e.m.p.a.'s have been obtained under some restrictions (existence of certain nontrivial common factors).

All these results, however, left open even the following natural question, which was one of the motivations for us to start this work. Given a family of two e.m.p.a.'s (or more concretely, of two irrational rotations of a circle) is it possible to realize it in a Cantor minimal system with exactly two ergodic invariant probability measures? We show here that the answer is positive even for arbitrary finite or countable families of e.m.p.a.'s. In particular, we show the following.

THEOREM 1: *Let $\{T_p : p \in \mathcal{I}\}$ be an arbitrary finite or countable collection of e.m.p.a.'s on nonatomic Lebesgue probability spaces (Y_p, ν_p). Let S be a Cantor minimal system such that the cardinality of the set $\mathcal{E}_S$ of all ergodic S-invariant Borel probability measures is at least the cardinality of $\mathcal{I}$. Then for any collection $\{\mu_p : p \in \mathcal{I}\}$ of distinct measures from $\mathcal{E}_S$ there is a Cantor minimal system S' with the same orbits as S such that, as a measure preserving system, (S', μ_p) is isomorphic to T_p for every $p \in \mathcal{I}$.*

Moreover, S' can be chosen strongly orbit equivalent to S (by the identity map) if and only if all finite topological factors of S are measure-theoretic factors of T_p for all $p \in \mathcal{I}$.

Probably the most interesting special case of Theorem 1 is the case when the cardinalities of $\mathcal{E}_S$ and of $\mathcal{I}$ are exactly the same. Then the theorem guarantees that, given any finite or countable list of (not necessarily distinct) ergodic transformations, one can find a Cantor minimal system whose collection of distinct ergodic measures realizes exactly the transformations in our list.

Recently, after a preliminary version of this paper was written, we received a preprint [4] of T. Downarowicz, whose results overlap with ours. For an arbitrary finite (but not for a countably infinite) family of e.m.p.a.'s, the positive answer to the realization problem follows also from [4]. The method of [4] is very different from ours, and the problem is considered not in the orbit theory setting. Roughly speaking, it is shown in [4] that the condition of minimality of S imposes only the same (and no additional) restrictions on realizability of arbitrary (finite, countable, or uncountable) families of e.m.p.a.'s (or endomorphisms) as the condition of aperiodicity of S.

When dealing with realizations of an (infinitely) countable family $\{T_p\}$ of e.m.p.a.'s, it is natural to expect *a priori* that the topology on the space of ergodic invariant measures for a homeomorphism S may impose additional restrictions for realizability (minimal, aperiodic, etc.). Because of that, the fact that there are no restrictions at all in the countable case looks, at least for us, a little bit counterintuitive, even modulo the positive result in the finite case. It should be noted, however, that a supporting argument for the positive countable result comes from [7], where the necessary entropy function condition for realizability of arbitrary families is established, and this necessary condition turns out to be void in the countable case. Based on that, the question whether or not there are any restrictions which are non-void in the countable case was asked in [4]. Specifically, Downarowicz conjectures that there are no restrictions in the countable case, and our work proves this conjecture.

In the present paper the topological models for the collections of e.m.p.a.'s will be constructed within the topological orbit equivalence classes of given homeomorphisms. In this connection, we need to mention some facts on measurable and topological orbit equivalence.

The notion of orbit equivalence for measure preserving and nonsingular transformations (and for countable transformation groups) was introduced and studied in the pioneering papers [8] and [9] of H. Dye. His fundamental theorem says that any two e.m.p.a.'s of nonatomic Lebesgue probability spaces are orbit equivalent. This result was obtained by H. Dye as a part of his study of some problems in the theory of von Neumann algebras. A far reaching generalization of Dye's theorem was given by W. Krieger [16]. He proved that two nonsingular ergodic automorphisms are orbit equivalent if and only if the von Neumann algebras (factors), corresponding to them via the Murray–von Neumann group-measure space construction, are isomorphic.

In the topological setting, the parallel theory of topological orbit equivalence, with C^*-algebras playing the role similar to the role of von Neumann algebras in the measurable setting, was initiated much later. It started in the work of A. Vershik [19], [20] on adic realizations of e.m.p.a.'s, and then was developed in the series of remarkable results of T. Giordano, R. Herman, I. Putnam, C. Skau, E. Glasner and B. Weiss [11], [13], [12]. In the relatively recent paper of N. Ormes [18], the orbit realization theorem (ORT) solves in a positive way the problem of constructing the topological models of an individual e.m.p.a. within the topological orbit equivalence class of a given Cantor minimal system. Both the Jewett–Krieger theorem and Dye's theorem are special cases of the results of

ORT. Also in [18] is the strong orbit realization theorem (SORT) which shows that, except for a necessary condition on periodic factors, a realization as in ORT can take place within a *strong* orbit equivalence class. In a strong orbit equivalence, the related orbit cocycle functions are required to have at most one point of discontinuity each. Thus SORT stands in stark contrast to a result of Boyle which states that two minimal systems related by continuous cocycles are conjugate or conjugate to the inverse of the other [2]. The results here demonstrate even more dramatically how time changes which are discontinuous only at a single point can alter the measure-theoretic properties of a minimal homeomorphism.

From the orbital point of view, the main results of [18] are the starting point of our study, and the technique of [18] is an essential ingredient in our proofs. Because of this, the reader is strongly advised to be familiar with [18]. Our goal is to extend the results of [18] from the case of a single e.m.p.a. to the case of arbitrary countable families of transformations.

The main step in [18] is, roughly speaking, an inductive construction of a nested sequence of multitowers (Kakutani skyscrapers) for an e.m.p.a. T and a compatible nested sequence for a given Cantor minimal system S with respect to some S-invariant ergodic measure μ. The floors of the multitowers for S are then rearranged so that the rearranged towers define a new homeomorphism S', for which μ is still invariant, and such that S' considered as a μ-preserving system is measure-theoretically isomorphic to T. This inductive construction of [18] is the prototype of the inductive construction in our paper, where, instead of a single measure μ, we are dealing with a family $\{\mu_p\}$ of measures.

The idea of the present paper (again, very roughly speaking) is the following. Given a finite family $\{T_1, T_2, \dots, T_m\}$ (instead of a single transformation T), and a Cantor minimal system S having at least m ergodic invariant Borel probability measures, we construct the nested multitowers, in a sense, simultaneously for all T_p's, $1 \leq p \leq m$, and do the rearrangements of their floors in the same way for each T_p. In the case of a countable family $\{T_1, T_2, \dots, T_m, \dots\}$, at every step $(m+1)$ of the inductive construction we bring, along with the measures $\mu_1, \dots, \mu_m$ considered at the previous m-th step, one more measure μ_{m+1}. This becomes possible because the mutual singularity of the distinct ergodic S-invariant measures μ_p's implies that, for each m, the range of the vector measure $(\mu_1, \dots, \mu_m)$ is the entire m-dimensional unit cube (this is a simple special case of the Lyapunov theorem on the range of the vector measure [17] though we do not need the full strength of that theorem). This, in turn, gives us enough flexibility to construct, on each step $(m+1)$ of the construction, the necessary multitowers in such a way that not only the previous measures $\mu_1, \dots, \mu_m$ can be taken care of with "better accuracy" than in the previous step, but in addition, the measure μ_{m+1} can join them.

The organization of this paper is the following. In §2 we introduce some necessary notions and give basic definitions. In §3 we state several lemmas which reduce the proof of the main theorem to the construction of certain sequences of multitowers for a homeomorphism S and for e.m.p.a.'s T_p. This section also contains the proofs of all these lemmas except the central Lemma 6. The proof of Lemma 6 is given in §4. Finally, §5 contains some concluding remarks.

2. Preliminaries

Now let us introduce some necessary definitions. For definitions that are not given here, see [18].

Let S be a minimal homeomorphism of the Cantor set X. Denote by $\mathcal{E}_S$ the set of all ergodic S-invariant Borel probability measures.

Let $\{(Y_p, T_p, \mathcal{B}_p, \nu_p) : p \in \mathcal{I}\}$ be a countable collection of e.m.p.a.'s of nonatomic Lebesgue probability spaces $\{Y_p\}$. We will primarily work with the case where $\mathcal{I}$ is countably infinite, as opposed to finite, and note when the situation is different (simpler) for the finite case. Let Y be the disjoint union of the Y_p's. The set Y becomes a measurable space if we introduce a σ-algebra $\mathcal{B}$ on it, namely the one generated by all σ-algebras $\mathcal{B}_p$'s on Y_p's. Each measure ν_p, $p \in \mathcal{I}$, can be pulled on Y as a measure concentrated on Y_p. With respect to each measure ν_p the space Y is a probability measure space. Let $T \colon Y \to Y$ be the transformation which is equal to T_p when restricted to Y_p. Then T is a measure preserving transformation of Y with respect to each ν_p. Let $\mathcal{F}_T = \{\nu_p : p \in \mathcal{I}\}$.

We will assume throughout that the cardinality of $\mathcal{E}_S$ is greater than or equal to the cardinality of $\mathcal{F}_T$, a necessary condition for the realization of the family of transformations in the orbit equivalence class of S. Enumerate any subcollection $\{\mu_p : p \in \mathcal{I}\}$ of distinct elements of $\mathcal{E}_S$. Our goal is to find S' orbit equivalent to S with (S', μ_p) measurably conjugate to (T, ν_p) for all $p \in \mathcal{I}$.

A **clopen multitower** for S is a collection of disjoint clopen sets

$$\sigma = \{A(i,j) |\ 1 \le i \le I, 0 \le j < H(i)\}$$

for some positive integers $I, H(1), H(2), \dots, H(I)$ where $SA(i,j) = A(i, j+1)$ for $0 \le j < (H(i) - 1)$ and $\bigcup_{i=1}^{I} \bigcup_{j=0}^{H(i)-1} A(i,j) = X$. We use the notation

$\sigma = \langle A(i,j); H(i); I\rangle$ to denote such a multitower. Define base$(\sigma) = \bigcup_{i=1}^{I} A(i,0)$ and top$(\sigma) = \bigcup_{i=1}^{I} A(i, H(i)-1)$. If Σ and σ are towers for S and every floor of σ is the union of floors of Σ we say that Σ **refines** σ. We say (σ, Σ) is a **nested pair**, or $\sigma \prec \Sigma$, if Σ refines σ and base$(\Sigma) \subseteq$ base(σ) (or equivalently, top$(\Sigma) \subseteq$ top(σ)). A sequence of towers $(\sigma_1, \sigma_2, \ldots)$ is called a **nested sequence** if $\sigma_m \prec \sigma_{m+1}$ for all m.

We will construct clopen multitowers for S, but also consider them measure-theoretically. For a clopen multitower σ for S and any $\mu \in \mathcal{M}_S$, we will consider the "architecture" of σ with respect to μ. That is, in addition to the number of towers I, and the heights of the towers $H(i)$, we will consider the measures of the floors $\mu(A(i,0))$.

A **measurable multitower** $\tau = \langle B(i,j); H(i); I\rangle$ for T is defined analogously to a clopen multitower for S, where the requirement that floors are clopen is replaced with the requirement that the floors are measurable (i.e., elements of $\mathcal{B}$), and the requirement that the union of floors be equal to X is replaced by the requirement that the union of floors is a set of full measure for every measure in $\mathcal{F}_T$.

Definition 2: Let $\sigma = \langle A(i,j); H(i); I\rangle$ be a clopen multitower for S and $\tau = \langle B(i,j); H'(i), I'\rangle$ a measurable multitower for T. Let $\mu \in \mathcal{E}_S$ and $\nu \in \mathcal{F}_T$. We say $(\sigma;\mu) \approx (\tau;\nu)$ if

(1) $\nu[\bigcup_{i=1}^{I} \bigcup_{j=0}^{H'(i)-1} B(i,j)] = 1$,
(2) $H(i) = H'(i)$ for $1 \le i \le I$,
(3) $\mu A(i,0) = \nu B(i,0)$ for $1 \le i \le I$.

Intuitively, the relation $(\sigma;\mu) \approx (\tau;\nu)$ means that σ and τ have the same architecture with respect to the chosen measures.

Definition 3: Let $(\sigma_1, \sigma_2, \ldots, \sigma_M)$ be a finite nested sequence of clopen multitowers for S. Set $\sigma_m = \langle A_m(i,j); H_m(i); I_m\rangle$.

Let $(\tau_1, \tau_2, \ldots, \tau_M)$ be a finite nested sequence of measurable multitowers for T. Set $\tau_m = \langle B_m(i,j); H'_m(i); I'_m\rangle$.

We say $(\sigma_1, \sigma_2, \ldots, \sigma_M)$ and $(\tau_1, \tau_2, \ldots, \tau_M)$ are **compatible with respect to measures** $\mu \in \mathcal{E}_T$ **and** $\nu \in \mathcal{F}_T$ if for all $1 \le m \le M$,

(1) $(\sigma_m;\mu) \approx (\tau_m;\nu)$,
(2) for all $1 \le i \le I_m$ and $1 \le k \le I_{m-1}$,
$\#\{A_m(i,j) |\ A_m(i,j) \subseteq A_{m-1}(k,0), 0 \le j < H_m(i)\}$
$= \#\{B_m(i,j) |\ B_m(i,j) \subseteq B_{m-1}(k,0), 0 \le j < H_m(i)\}$,
(3) for all $1 \le i \le I_m$ and $1 \le k \le I_{m-1}$,
$A_m(i,0) \subseteq A_{m-1}(k,0) \iff B_m(i,0) \subseteq B_{m-1}(k,0)$,

(4) for all $1 \leq i \leq I_m$ and $1 \leq k \leq I_{m-1}$,
$A_m(i, H_m(i)-1) \subseteq A_{m-1}(k, H_{m-1}(k)-1)$
$\iff B_m(i, H_m(i)-1) \subseteq B_{m-1}(k, H_{m-1}(k)-1)$.

Notation 4: We will use the notation

$$(\sigma_1, \sigma_2, \ldots, \sigma_M; \mu) \approx (\tau_1, \tau_2, \ldots, \tau_M; \nu)$$

if the two sequences of towers are compatible with respect to measures μ and ν.

Note that conditions (2)–(4) in Definition 3 are void when $M = 1$, so there is no conflict between this notation and Definition 2.

The point of the definition of compatible sequences of multitowers is that with this condition, we may define a sequence of bijections of floors of towers. This is demonstrated in the lemmas of the following section.

3. Reduction lemmas

3.1. Correspondence of Floors. Suppose $(\sigma_1, \sigma_2, \ldots, \sigma_M)$ and $(\tau_1, \tau_2, \ldots, \tau_M)$ are two finite nested sequences of multitowers for S and T which are compatible with respect to $\mu \in \mathcal{E}_T$ and $\nu \in \mathcal{F}_T$. We may then define a collection of bijections $\{h_m : 1 \leq m \leq M\}$ where h_m is a bijection from the set of floors of σ_m to the set of floors of τ_m intersected with a set of ν measure 1 in the following way.

Define $h_1(A_1(i,j)) = B_1(i,j)$.

Then for $m > 1$, we first define $h_m(A_m(i,0)) = B_m(i,0)$. Condition (3) of compatibility insures

$$A_m(i,0) \subset A_{m-1}(k,0) \Longrightarrow h_m(A_m(i,0)) \subset h_{m-1}(A_{m-1}(k,0))$$

so that the maps h_m and h_{m-1} respect the subset relation.

We define $h_m(A_m(i,j))$ recursively for $j > 0$. Having defined $h_m(A_m(i,\hat{\jmath}))$, for $\hat{\jmath} < j$, we define $h_m(A_m(i,j))$ to be the floor $B_m(i,j')$ where j' is the minimum floor height such that $B_m(i,j')$ is not already the image of a floor and $B_m(i,j') \subset h_{m-1}(A_{m-1}(k,l))$ where (k,l) is the index with $A_m(i,j) \subset A_{m-1}(k,l)$. Condition (2) insures that there are exactly the right number of floors of each type to make this assignment.

In particular, we have recursively defined a collection of bijections which

(1) respects the subset relation, i.e., if $A_m(i,j) \subset A_{m-1}(k,l)$ then $h_m(A_m(i,j)) \subset h_{m-1}(A_{m-1}(k,l))$,

(2) is measure-preserving, i.e., $\nu(h_m(A_m(i,j))) = \mu(A_m(i,j))$,
(3) respects columns, i.e., for all i,j, $h_m(A_m(i,j)) = B_m(i,l)$ for some l,
(4) respects bases, i.e., for all i, $h_m(A_m(i,0)) = B_m(i,0)$,
(5) respects tops, i.e., for all i, $h_m(A_m(i,H_m(i)-1)) = B_m(i,H_m(i)-1)$.

Furthermore, if $A_m(i,j)$ is a non-top floor of σ_m then $h_m(A_m(i,j))$ is a non-top floor of τ_m. Therefore, $h_m^{-1}Th_m(A_m(i,j)) = A_m(i,j')$ for some $0 \le j' < H_m(i)$. Set $n_m(i,j) = j' - j$.

We have insured in our definition of h_m that if $A_m(i,j) \subset A_{m-1}(k,l)$ and $A_{m-1}(k,l)$ is a non-top floor, then $n_{m-1}(k,l) = n_m(i,j)$.

The point that the above maps h_m and n_m are recursively defined is more than a remark. It is important to our construction that we can fix all of the above for the first m multitowers, then add another $(m+1)$st multitower and extend all of the maps above.

As we see below, if we replace the finite sequences of multitowers with an infinite sequence of compatible multitowers satisfying separation properties, then we will be able to conclude that e.m.p.a.'s (S,μ) and (T,ν) are measurably conjugate.

LEMMA 5 (Reduction 1): *Suppose there exists an infinite sequence of nested clopen multitowers $\sigma_1 \prec \sigma_2 \prec \cdots$ for S, and an infinite sequence of nested measurable multitowers $\tau_1 \prec \tau_2 \prec \cdots$ for T such that*

(1) $(\sigma_1, \sigma_2, \ldots, \sigma_m; \mu_p) \approx (\tau_1, \tau_2, \ldots, \tau_m; \nu_p)$ *for all $m, p \ge 1$,*
(2) $\bigcap_m \operatorname{base}(\sigma_m)$ *is one point set $\{x_0\}$ and $\bigvee_m \sigma_m$ generates the topology of X,*
(3) *the partitions $\{\tau_m\}_{m\in\mathbb{N}}$ separate points in Y on a set $Y_0 \subset Y$ with $\nu_p(Y_0) = 1$ for all p.*

Then there is a minimal homeomorphism S' such that S and S' are strongly orbit equivalent by the identity map and, for every $\mu_p \in \mathcal{E}_S$, (S', μ_p) is measurably conjugate to (T_p, ν_p).

Proof: Given the above hypotheses, define the countable collection of bijections h_m as in Section 3.1. Since the sequence of clopen towers $\{\sigma_m\}$ separates points, every point $x \in X$ is uniquely identified according to the sequence of floors in $\{\sigma_m : m \ge 1\}$ which contain it.

Note that $\bigcap_m \operatorname{top}(\sigma_m) = \{S^{-1}(x_0)\}$. If $x \ne S^{-1}(x_0)$, then there is a tower σ_m such that $x \notin \operatorname{top}(\sigma_m)$. Let $A_m(i,j)$ denote the floor of σ_m containing x. Define $S'(x) = S^{n_m(i,j)}(x)$ where n_m is as described in the text preceding the statement of this lemma. Because the maps n_m respect subsets, the definition of $S'(x)$ does not depend on the choice of m. If $x = S^{-1}(x_0)$, set $S'(x) = S(x)$. The function $n(x)$ which satisfies $S'(x) = S^{n(x)}(x)$ has at most one point of discontinuity (namely, $S^{-1}(x_0)$) since $n(x)$ is constant on any (clopen) floor of σ_m which is not a subset of $\mathrm{top}(\sigma_m)$. The same holds for the function $m(x)$ satisfying $S(x) = (S')^{m(x)}(x)$.

That S' is continuous on $X \setminus \{S^{-1}(x_0)\}$ follows from the fact that n is constant on clopen sets (floors of σ_m). That S' is continuous at $S^{-1}(x_0)$ follows from the fact that $S'(\mathrm{top}(\sigma_m)) = \mathrm{base}(\sigma_m) \ni x_0$ and $\bigcap_m \mathrm{base}(\sigma_m) = \{x_0\}$. The minimality of S' follows from the fact that S and S' have the same orbits, and S is minimal. Therefore, S and S' are minimal homeomorphisms of the Cantor set, strongly orbit equivalent by the identity map.

Now fix $\mu_p \in \mathcal{E}_S$. Since the multitowers $\{\sigma_m\}$ separate points in X and $\{\tau_m\}$ separates points on a set of ν_p-measure 1, the sequence of maps $\{h_m\}$ can be extended to a measure-space isomorphism $h : (X, \mu_p) \to (Y, \nu_p)$. For every non-top floor $A(i,j) \in \sigma_m$, we have $S'(A(i,j)) = h^{-1}Th(A(i,j))$. We also have $\mu_p(\bigcap_m \mathrm{top}(\sigma_m)) = 0$. Therefore the map h gives a measurable conjugacy between (S', μ_p) and (T_p, ν_p). ∎

Our task then is to construct multitowers as in the previous lemma. This will be done by an inductive process, which is essentially described in Lemma 6. This lemma, as well as Proposition 7 below, will be used in the proof of the realization theorem for finite or countably infinite families of e.m.p.a.'s. In the case of a finite family $\{T_p\}$ the inductive process is slightly different (simpler), since there is no need to bring an additional measure μ_{m+1} on the $(m+1)$-st step, if $m+1$ is bigger than the number of the automorphisms in the family $\{T_p\}$. Because of this, some insignificant changes (replacement of $m+1$ by m) are needed in the statement of Lemma 6 in the finite case. We leave these details to the reader.

LEMMA 6 (Reduction 2): *Suppose every finite (periodic) factor of S is a finite (periodic) factor of T_p for all $p \in \mathcal{I}$.*

Let $\epsilon > 0$. Let $\mathcal{P}$ be a clopen partition of X, and let $x_0 \in X$. Let $\mathcal{Q}$ be a finite partition of $\bigcup_{p=1}^{m+1} Y_p$ which is measurable with respect to ν_p for $1 \le p \le m+1$.

Suppose there exist a clopen multitower σ for S, and τ a measurable multitower for T such that $(\sigma; \mu_p) \approx (\tau; \nu_p)$ for every $1 \le p \le m$, and $x_0 \in \mathrm{base}(\sigma)$. Then there exists a clopen multitower Σ for S and measurable multitowers τ' and $\mathcal{T}$ for T such that

(1) $(\sigma, \Sigma; \mu_p) \approx (\tau', \mathcal{T}; \nu_p)$ *for* $1 \le p \le m+1$,

(2) $\|\tau - \tau'\|_{\nu_p} < \epsilon$ *for* $p = 1, 2, \ldots, m$,

(3) $x_0 \in \text{base}(\Sigma)$, $diam(\text{base}(\Sigma)) < \epsilon$,
(4) Σ *refines* $\mathcal{P}$,
(5) $\mathcal{T}$ *refines* $\mathcal{Q}$ *on a set* E *with* $\nu_p(E) > 1 - \epsilon$ *for* $p = 1, 2, \ldots, m+1$.

Before we turn our attention to the proof of the lemma above, let us indicate how the realization theorem follows from it.

PROPOSITION 7: *Lemma 6 implies Theorem 1.*

Proof: Let S and T be as in the hypotheses of Theorem 1, and assume Lemma 6 is true.

First we explain that if our interest is orbit equivalence, as opposed to strong orbit equivalence, we may assume without loss of generality that S has no finite factors, and thus assume that S and T satisfy the hypotheses of Lemma 6. By Lemma 7.1 of [18], there is a unital ordered group $\mathcal{G} = (G, G_+, u)$ where (G, G_+) is an acyclic simple dimension group, $\mathcal{G}/Inf(\mathcal{G}) \cong \mathcal{G}^S/Inf(\mathcal{G}^S)$ and G has trivial rational subgroup. Further, by [13], there is a minimal homeomorphism $\widehat{S}$ of the Cantor set with $\mathcal{G}^{\widehat{S}} \cong \mathcal{G}$. By [11], $\widehat{S}$ and S are orbit equivalent. Since $\widehat{S}$ and S are orbit equivalent, $\mathcal{M}_S$ and $\mathcal{M}_{\widehat{S}}$ are affinely homeomorphic. Thus, by replacing S with $\widehat{S}$ we have the hypotheses of both Theorem 1 and of Lemma 6 and this replacement does not change the orbit equivalence class in which we are working.

For the strong orbit equivalence statement of Theorem 1, we note that the necessity of the condition on finite factors for S and T_p is already known (see [18], p. 110), so we only need to prove the sufficiency part.

Now let $\{\epsilon_m\}$ be a summable sequence of positive numbers. Let $\{\mathcal{P}_m\}_{m\in\mathbb{N}}$ be a refining sequence of (finite) clopen partitions in X which generate the topology. Let $\mathcal{P}_0$ and σ_0 be a trivial partition of X and multitower for S, respectively. Let $\{\mathcal{Q}_m\}_{m\in\mathbb{N}}$ be a refining sequence of finite measurable partitions of Y which separate points on a set of full measure in Y with respect to all $\nu \in \mathcal{F}_T$. Let $\mathcal{Q}_0$ and $\tau_0^{(0)}$ be a trivial partition of Y and multitower for T, respectively. Assume that for some $M \geq 0$, we have a finite nested sequence of clopen multitowers $\sigma_1 \prec \sigma_2 \prec \cdots \prec \sigma_M$ for S and a finite nested sequence of multitowers $\tau_1^{(M)} \prec \tau_2^{(M)} \prec \cdots \prec \tau_M^{(M)}$ satisfying

(1) $(\sigma_1, \sigma_2, \ldots, \sigma_M; \mu_p) \approx (\tau_1^{(M)}, \tau_2^{(M)}, \ldots \tau_M^{(M)}; \nu_p)$ for $1 \leq p \leq M$,
(2) $x_0 \in \bigcap_{p=1}^M \text{base}(\sigma_p)$, $diam(\bigcap_{p=1}^M \text{base}(\sigma_p)) < \epsilon_M$, σ_M refines $\mathcal{P}_M$,
(3) $\tau_M^{(M)}$ refines $\mathcal{P}_M$ on a set E_M with $\nu_p(E_M) > 1 - \epsilon_M$.

Now apply Lemma 6 with $\epsilon = \epsilon_M$, $\mathcal{P} = \mathcal{P}_{M+1}$, $\mathcal{Q} = \mathcal{Q}_{M+1}$, $\sigma = \sigma_M$, $\tau = \tau_M^{(M)}$ and the collections of measures $\mu_1, \mu_2, \ldots, \mu_{M+1}$ and $\nu_1, \nu_2, \ldots, \nu_{M+1}$.

Set $\sigma_{M+1} = \Sigma$, $\tau_{M+1}^{(M+1)} = \mathcal{T}$ and $\tau_M^{(M+1)} = \tau'$.

We define the multitowers $\tau_m^{(M+1)}$ for $0 \le m < M$ in the following way. When intersected with $\bigcup_{p=1}^M Y_p$, every floor F of $\tau_{M-1}^{(M)}$ is a union over some index set of floors of $\tau_M^{(M)}$. We let F' be the union over the same index set of floors of $\tau_{M-1}^{(M+1)}$, but now intersected with $\bigcup_{p=1}^{M+1} Y_p$. Thus we obtain a new multitower $\tau_{M-1}^{(M+1)}$ for T where

$$(\tau_{M-1}^{(M+1)}; \nu_p) \approx (\tau_{M-1}^{(M)}; \nu_p)$$

and

$$||\tau_{M-1}^{(M)} - \tau_{M-1}^{(M+1)}||_{\nu_p} < \epsilon_M$$

for $1 \le p \le M$. Proceeding in this way, we define $\tau_m^{(M+1)}$ for $m = M-1$, $M-2, \ldots, 1$.

Clearly $\sigma_1 \prec \sigma_2 \prec \cdots$ forms an infinite sequence of nested clopen multitowers where $\{x_0\} = \bigcap_{m=1}^\infty \mathrm{base}(\sigma_m)$, and $\bigvee \sigma_m$ generates the topology of X.

Fix $p \ge 1$. For $M \ge m, p$, we have

$$||\tau_m^{(M)} - \tau_m^{(M+1)}||_{\nu_p} < \epsilon_M.$$

Since $\sum_{M \ge 0} \epsilon_M < \infty$ the Borel–Cantelli Lemma implies that there are ν_p-measurable multitowers $\tau_m = \lim_{M\to\infty} \tau_m^{(M)}$. Since $(\sigma_m; \mu_p) \approx (\tau_m^{(M)}; \nu_p)$ for $M \ge m, p$, it is also true that $(\sigma_m; \mu_p) \approx (\tau_m; \nu_p)$. Similarly,

$$(\sigma_1, \sigma_2, \ldots, \sigma_m; \mu_p) \approx (\tau_1, \tau_2, \ldots, \tau_m; \nu_p).$$

If $m \ge p$, the multitower $\tau_m^{(m)}$ refines $\mathcal{Q}_m$ on a set E_m with $\nu_p(E_m) > 1 - \epsilon_m$. Since $||\tau_m - \tau_m^{(m)}||_{\nu_p} < \sum_{M \ge m+1} \epsilon_M$, we have that τ_m refines $\mathcal{P}_m$ on a set F_m where $\nu_p(F_m) > 1 - \sum_{M \ge m} \epsilon_M$. Therefore, as partitions, the collection of multitowers $\{\tau_m\}$ separates points on a set of full ν_p-measure for every $p \ge 1$. Therefore, the conditions of Lemma 5 are satisfied and Theorem 1 holds. ∎

It remains then to verify Lemma 6. Its proof follows to a great extent the scheme of the proof of Lemma 5.1 of [18]. We reproduce here the relevant definitions of [18] and include the proofs of all statements that are not precisely the same as those in [18].

4. Proof of Lemma 6

Let us assume that the multitowers

$$\sigma = \langle a(k,l); h(k); K\rangle \quad \text{and} \quad \tau = \langle b(k,l); h(k); K\rangle$$

are as in the hypothesis of Lemma 6. We must construct three multitowers Σ, τ' and $\mathcal{T}$. To do this requires us to construct several intermediate multitowers.

4.1. Step 1 — Getting Started. First we note that, without loss of generality, we may assume that the heights $h(1), h(2), \ldots, h(K)$ are relatively prime, i.e., $\gcd(h(1), h(2), \ldots, h(K)) = 1$. Assume that we can prove Lemma 6 in this setting. Now if $\gcd(h(i)) = g \neq 1$, then we consider the multitowers $\sigma_g = \langle a_g(k,l); h(k)/g; K\rangle$ and $\tau_g = \langle b_g(k,l); h(k)/g; K\rangle$ where $a_g(k,l) = a(k, gl)$ and $b_g(k,l) = b(k,gl)$. These multitowers σ_g, τ_g are multitowers for the maps S^g and T^g on the spaces $X_g = \bigcup a_g(k,l)$ and $Y_g = \bigcup b_g(k,l)$ with measures $g\mu_p$ and $g\nu_p$. Note that, in particular, we have $x_0 \in \text{base}(\sigma_g)$. In this situation we may set $\epsilon_g = \epsilon/g$, $\mathcal{P}_g = \bigvee_{i=0}^{g-1} S^i\mathcal{P}$, and $\mathcal{Q}_g = \bigvee_{i=0}^{g-1} T^i\mathcal{Q}$ and apply the (relatively prime version of the) lemma to obtain multitowers Σ_g, τ_g' and $\mathcal{T}_g$. We can then define a multitower Σ with base equal to the base of Σ_g and floors equal to S^i-images of floors of Σ_g for $i = 0, 1, \ldots, g-1$. We define τ' and $\mathcal{T}$ similarly using T^i-images of floors of τ_g' and $\mathcal{T}_g$. The towers Σ, τ' and $\mathcal{T}$ satisfy the desired conditions. Thus from here forward we will assume that $\gcd(h(1), h(2), \ldots, h(K)) = 1$.

Next we prove a simple lemma to begin the construction.

Lemma 8: *Given any integer $Z > 0$ we can create a clopen multitower $\Sigma_1 = \langle A(i,j); \widehat{H}(i); \widehat{I}\rangle$ for S where $\Sigma_1 \succ \sigma$ and Σ_1 satisfies the following properties:*

(1) $x_0 \in \text{base}(\Sigma_1)$,
(2) $diam(\text{base}(\Sigma_1)) < \epsilon$,
(3) $\text{base}(\Sigma_1)$ *is a subset of the floor of* σ *which contains* x_0,
(4) $\top(\Sigma_1)$ *is a subset of the floor of* σ *which contains* $S^{-1}(x_0)$,
(5) $\widehat{H}(i) > Z$ *for all* i,
(6) Σ_1 *refines* $\mathcal{P}$.

To achieve all of the above, it is only necessary to select a clopen multitower whose base is a clopen neighborhood of x_0 with sufficiently small diameter and afterwards to refine the columns so that the resulting multitower refines both σ and $\mathcal{P}$. The value of Z which we will use in this particular construction will be specified later after several definitions and notations are established (Section 4.4.1). For now we remark that the choice is based on the Ergodic Theorem applied to indicator functions of the floors in the towers σ and τ and measures μ_p, ν_p with $1 \leq p \leq m+1$. Since we currently have all of these elements at our disposal, let us assume that we have created $\Sigma_1 = \langle A(i,j); \widehat{H}(i); \widehat{I}\rangle$.

4.2. Step 2 — The Copying Lemma. Next we create a multitower $\mathcal{T}_1$ for T satisfying $(\mathcal{T}_1;\nu_p) \approx (\Sigma_1;\mu_p)$ for $p = 1, 2, \ldots, m+1$. This requires a version of the Alpern Multitower Lemma below [1]. See [10] for a simple proof which follows Kakutani's idea of the proof of the the classical Rokhlin Lemma.

Lemma 9 (Alpern Multitower Lemma): *Let (Y, T, ν) be an arbitrary e.m.p.a. of a nonatomic Lebesgue probability space.*

Let $H(1), H(2), \ldots, H(I)$ be a set of positive integers with

$$\gcd\{H(i) : 1 \le i \le I\} = g,$$

and let $\alpha(1), \alpha(2), \ldots, \alpha(I)$ be positive real numbers with $\sum_{i=1}^{I} H(i)\alpha(i) = 1$. If T has a finite factor of cardinality g (i.e., a factor consisting of one orbit of cardinality g), then there is a multitower $\mathcal{T} = \langle B(i,j); H(i); I\rangle$ for T with $\nu(B(i,0)) = \alpha_i$.

Our statement of Alpern's Multitower Lemma is slightly different from the standard one (in which it is assumed that $\gcd\{H(i) : 1 \le i \le I\} = 1$). We need a more general version despite the fact that our argument in Step 1 allows us to assume that the heights h_i's of the original tower σ are relatively prime. At the moment we are copying the tower heights from Σ_1, and since we allow periodic factors for S, we are not able to assume that these new tower heights are also relatively prime. Nevertheless, it is a simple matter to show that our version of the Alpern lemma follows from the standard one.

Let $g = \gcd\{H(i)\}$. Since the orbit of period g is a factor of T, there is a ν-measurable set B with $\nu(T^iB \cap T^jB) = 0$ for $0 \le i < j < g$ and $\nu(\bigcup_{i=0}^{g-1} T^iB) = 1$. Apply the usual Multitower Lemma to the induced system $(B, T^g, g\nu)$ and $H(i)/g$ to produce a multitower $\langle B(i,j); H(i)/g; I\rangle$ for T^g. Then set $\widehat{B}(i,j) = T^jB(i,0)$ for $0 \le j < H(i)$; we have that $\langle \widehat{B}(i,j); H(i); I\rangle$ is the desired multitower for T.

In our setting, we may apply the above lemma to each of the e.m.p.a.'s (Y_p, T_p, ν_p) for $1 \le p \le m+1$, with $I = \widehat{I}$, $H(i) = \widehat{H}(i)$ and $\alpha_i = \mu_p(A(i,0))$. Then by unioning the floors of the individual $m+1$ different multitowers together, we can create a multitower $\mathcal{T}_1$ for T satisfying $(\mathcal{T}_1;\nu_p) \approx (\Sigma_1;\mu_p)$ for $1 \le p \le m+1$. We may assume $(\mathcal{T}_1;\nu_p)$ is the trivial multitower for $p > m+1$.

Next we refine the towers of $\mathcal{T}_1$ by the partition $\mathcal{Q}$ and by τ. Call the resulting multitower $\mathcal{T}_2 = \langle B(i,j); H(i); I\rangle$. Note that although $\mathcal{T}_2$ refines τ as a partition, the pair $(\tau, \mathcal{T}_2)$ is not nested since the base of $\mathcal{T}_2$ need not be a subset of the base of τ.

4.3. Step 3 — Labellings. We now consider two labellings on $\mathcal{T}_2$. For us, a **labelling** $\mathcal{L}$ on a multitower is a map from the set of floors of the multitower to a finite subset of $\mathbb{N} \times \mathbb{N}$. The labellings we have in mind are generally those defined by a refining multitower. For example, for $\tau = \langle b(k,l); h(k); K\rangle$, we have a labelling $\mathcal{L}$ on $\mathcal{T}_2$ where $\mathcal{L}$ is the map from floors of $\mathcal{T}_2$ to pairs of the form $\{(k,l) : 1 \leq k \leq K, 0 \leq l < h(k)\}$ where $\mathcal{L}(B(i,j)) = (k,l)$ if $B(i,j) \subset b(k,l)$.

The point of defining labellings is to work in the other direction. That is, given the multitower $\mathcal{T}_2$ we wish to define a labelling which gives rise to a multitower τ' with $\tau' \prec \mathcal{T}_2$, $(\tau'; \nu_p) \approx (\sigma; \mu_p)$ and $\|\tau' - \tau\|_{\nu_p} < \epsilon$ for $1 \leq p \leq m$. In this vein, we define (h,β)-labellings ([18], Def. 5.2, p. 120).

Definition 10: Let $\mathcal{T} = \langle B(i,j); H(i); I\rangle$ be a multitower for an e.m.p.a. (Y,T,ν). Let $h = (h(1), h(2), \ldots, h(K))$ be an element of $\mathbb{N}^K$ and let $\beta = (\beta(1), \beta(2), \ldots, \beta(K))$ be a K-tuple of positive real numbers such that $\sum_{k=1}^{K} h(k)\beta(k) = 1$. We say a labelling $\mathcal{L}$ of $\mathcal{T}$ is an (h,β)-**labelling** with respect to ν if it has the following properties:

(1) $Image(\mathcal{L}) = \{(k,l) \in \mathbb{N}^2 |\ 1 \leq k \leq K, 0 \leq l < h(k)\}$,

(2) for $0 \leq j < (H(i) - 1)$,
if $\mathcal{L}(B(i,j)) = (k,l)$ with $l < (h(k) - 1)$ then $\mathcal{L}(B(i,j+1)) = (k, l+1)$,
if $\mathcal{L}(B(i,j)) = (k, h(k) - 1)$ then $\mathcal{L}(B(i,j+1)) = (k', 0)$ for some k',

(3) $\nu(\mathcal{L}^{-1}\{(k,l)\}) = \beta(k)$ for all (k,l).

Thus for any $1 \leq p \leq m$, setting $\beta_p(k) = \nu_p(b(k,0))$, the labelling $\mathcal{L}$ defined on $\mathcal{T}_2$ via the relationship $\tau \prec \mathcal{T}_2$ is an (h, β_p)-labelling with respect to ν_p. Conversely, to define τ' with $(\tau'; \nu_p) \approx (\tau; \nu_p)$ we need to define a new labelling on $\mathcal{T}_2$ which is an (h, β_p)-labelling with respect to ν_p. In addition, we would like $(\tau', \mathcal{T}_2)$ to be a nested pair. This motivates the following definition.

Definition 11: An (h,β)-labelling on a multitower $\mathcal{T} = \langle B(i,j); H(i); I\rangle$ is **nested** if $\mathcal{L}(B(i,0)) \in \{(k,0) |\ k \in \mathbb{N}\}$ for all i.

Recall that the important feature of the pair of multitowers for T that we need to arrange is that they be compatible with the pair of S multitowers. In this vein we consider a second labelling on $\mathcal{T}_2$ which we call $\mathcal{L}'$. This one is induced by the relationships $\sigma \prec \Sigma_1$, $(\Sigma_1; \mu_p) \approx (\mathcal{T}_1; \nu_p)$ and $\mathcal{T}_1 \prec \mathcal{T}_2$ for $1 \leq p \leq m$. That is, the relation $\sigma \prec \Sigma_1$ gives rise to a labelling $\mathcal{K}$ on Σ_1. Since $(\Sigma_1; \mu_p) \approx (\mathcal{T}_1; \nu_p)$ there is a measure-preserving bijection of floors (Section 3.1) which pushes the labelling $\mathcal{K}$ onto a labelling on $\mathcal{T}_1$. Finally, if a floor of $\mathcal{T}_1$ has a label (k,l), then we may assign that same label to all subset floors in $\mathcal{T}_2$. This defines a nested labelling $\mathcal{L}'$. For $1 \le p \le m$, since $(\sigma; \mu_p) \approx (\tau; \nu_p)$, the labelling $\mathcal{L}'$ is also an (h, β)-labelling of $\mathcal{T}_2$ with respect to ν_p with $\beta(k) = \nu_p(b(k, 0))$.

With the assistance of the following notation, we are able to restate our goal in terms of labellings.

Notation 12: For the labelling $\mathcal{L}$ on $\mathcal{T}_2 = \langle B(i,j); H(i); I\rangle$, let $N_k[m,n)_i$ be the number of floors in the ith column whose height index is between m and $(n-1)$ and whose label is $(k, 0)$, i.e.,

$$N_k[m,n)_i = \#\{B(i,j) | calL(B(i,j)) = (k,0) \text{ and } j \in [m,n)\}.$$

We will use the notation $N_k'[m,n)_i$ and $N_k''[m,n)_i$ similarly for labellings $\mathcal{L}'$ and $\mathcal{L}''$ of $\mathcal{T}_2$.

Definition 13: For two labellings of the same multitower $\mathcal{L}$ and $\mathcal{L}'$, let $||\mathcal{L} - \mathcal{L}'||_\nu$ denote the ν-measure of the collection of floors which have different images under $\mathcal{L}$ and $\mathcal{L}'$.

We seek a labelling $\mathcal{L}''$ of $\mathcal{T}_2 = \langle B(i,j); H(i); I\rangle$ such that

(1) for $1 \le p \le m$, $\mathcal{L}''$ is an (h, β_p)-labelling with respect to ν_p where $\beta_p(k) = \nu_p(b(k,0))$,
(2) $N_k''[0, H(i))_i = N_k'[0, H(i))_i$ for all i and k,
(3) $\mathcal{L}''(B(i,0)) = \mathcal{L}'(B(i,0))$ for all i (therefore insuring that the labelling is nested),
(4) $\mathcal{L}''(B(i, H(i)-1)) = \mathcal{L}'(B(i, H(i)-1))$ for all i,
(5) $||\mathcal{L}'' - \mathcal{L}||_{\nu_p} < \epsilon/2$ for $1 \le p \le m$.

4.4. Step 4 — Relabelling. To prove that a labelling of the required form exists, we will invoke the Ergodic Theorem. If the columns of $\mathcal{T}_2$ are sufficiently tall, the Ergodic Theorem gives us control over the label frequencies.

Definition 14: Let $\mathcal{T} = \langle B(i,j); H(i); I\rangle$ be a measurable multitower for an e.m.p.a. (Y, T, ν). Let $\mathcal{L}$ be an (h, β)-labelling of $\mathcal{T}$ for some (h, β). For all i, let $C(i) = \bigcup_{j=0}^{H(i)-1} B(i,j)$. We say $\mathcal{L}$ satisfies the δ-**frequency condition** with respect to ν if there is a collection of columns $G = \{C(i_1), C(i_2), \ldots, C(i_L)\}$ such that $\nu(G) > (1-\delta)$ and, for $C(i) \in G$,

$$2\delta H(i) \le m \le H(i) \quad \text{implies} \quad \left|\frac{1}{m} N_k[0,m)_i - \beta(k)\right| < \delta \quad \text{for all } k.$$

The δ-frequency condition is guaranteed in a multitower with sufficiently tall columns.

LEMMA 15 ([18], Lemma 5.6, p. 124): *Let (Y, T, ν) be an arbitrary e.m.p.a. and let τ be a measurable multitower for T and let $0 < \delta < \frac{1}{2}$ be given. There is an integer $Z_2(\delta, T, \nu)$ such that if a multitower $\mathcal{T}$ refines the multitower τ and the column heights of $\mathcal{T}$ all exceed $Z_2(\delta, T, \nu)$, then the labelling on $\mathcal{T}$ defined by τ satisfies the δ-frequency condition with respect to ν.*

See [18] for the proof, an application of the Pointwise Ergodic Theorem.

Given any $\delta > 0$, for each $1 \leq p \leq m$, we can apply the above lemma to (X, S, μ_p) and σ and to (X, T, ν_p) and τ. By taking the maximum Z_2 value, we can insure the conclusion of the lemma for all of these measures simultaneously. Note that this value of Z_2 depends only on δ, σ, τ, and the measures μ_p and ν_p where $1 \leq p \leq m$.

Next we prove the main lemma for the proof of our main theorem, the Labelling Lemma, which establishes the existence of a labelling satisfying the desired properties. This is the finite family version of Lemma 5.5 of [18]. We reproduce the proof of that lemma with the appropriate changes. In fact, only few changes are needed.

Recall that we are working under the assumption that the greatest common divisor of $h(1), h(2), \ldots, h(K)$ is 1 (Step 1).

LEMMA 16 (Labelling Lemma): *Let $h = (h(1), h(2), \ldots, h(K))$ be such that $\gcd(h(1), h(2), \ldots, h(K)) = 1$ and for $1 \leq p \leq m$, let*

$$\beta_p = (\beta_p(1), \beta_p(2), \ldots, \beta_p(K))$$

be positive real numbers such that $\sum_k h(k)\beta_p(k) = 1$.

Let $\mathcal{T} = \langle B(i,j); H(i); I \rangle$ be a measurable tower for T and let $\mathcal{L}, \mathcal{L}'$ be (h, β_p)-labellings of $\mathcal{T}$ with respect to ν_p.

For any $0 < \epsilon < 1$, there exist numbers $0 < \delta < 1/2$ and $Z_1(\delta)$ such that if $H(i) \geq Z_1(\delta)$ for all i, $\mathcal{L}$, $\mathcal{L}'$ satisfy the δ-frequency condition with respect to all measures ν_p, $1 \leq p \leq m$, and $\mathcal{L}'$ is a nested labelling, then there exists a nested labelling $\mathcal{L}''$ of $\mathcal{T}$ such that

(1) *$\mathcal{L}''$ is an (h, β_p)-labelling with respect to ν_p,*
(2) *$N''_k[0, H'(i))_i = N'_k[0, H'(i))_i$ for all i and k,*
(3) *$\mathcal{L}''(B(i, 0)) = \mathcal{L}'(B(i, 0))$ for all i,*
(4) *$\mathcal{L}''(B(i, H'(i) - 1)) = \mathcal{L}'(B(i, H'(i) - 1))$ for all i,*
(5) *$||\mathcal{L}'' - \mathcal{L}||_{\nu_p} < \epsilon/2$ for $p = 1, 2, \ldots, m$.*

Proof: Let J be an integer such that for any $j \geq J$, there exist non-negative integers $m_1, m_2, \ldots, m_K$ where $j = \sum_{k=1}^{K} m_k h(k)$. Choose δ and $Z_1(\delta)$ such that:

$$0 < \delta < \epsilon\beta_p(k)/6 \text{ for all } k, p, \quad \text{and} \quad Z_1(\delta) > (J + 3\max h(k))/\delta.$$

Assume $\mathcal{T} = \langle B(i,j); H(i); I\rangle$ is a multitower with $H(i) \geq Z_1(\delta)$ for all i. Let $\mathcal{L}$ and $\mathcal{L}'$ be two labellings of $\mathcal{T}$ which satisfy the hypotheses of the lemma.

Because the labellings $\mathcal{L}, \mathcal{L}'$ both satisfy the δ-frequency condition with respect to all measures ν_p for $1 \leq p \leq m$, there is a collection of columns G_p, G_p', as in Definition 14, for each $1 \leq p \leq m$. Let $G = \bigcup_{p=1}^m (G_p \cap G_p')$. From here the proof from [18] essentially carries over.

From the definition of G, if $C(i) \in G$ then there is a p, $1 \leq p \leq m$ such that for all k and for all m, $2\delta H(i) \leq m \leq H(i)$ we have

$$\left|\frac{1}{m}N_k[0,m)_i - \beta_p(k)\right| < \delta \quad \text{and} \quad \left|\frac{1}{m}N_k'[0,m)_i - \beta_p(k)\right| < \delta.$$

It is the fact that $\frac{1}{m}N_k[0,m)_i$ and $\frac{1}{m}N_k'[0,m)_i$ are close to one another that is important here, not that either is close to $\beta_p(k)$. This is why the proof for one measure carries over so easily to a finite collection of measures. Also take note that $\nu_p(\bigcup_{C(i)\in G} C(i)) > (1-2\delta)$ for any $1 \leq p \leq m$.

For any floor $B(i,j)$ where $C(i) \notin G$, let $\mathcal{L}''(B(i,j)) = \mathcal{L}'(B(i,j))$. In sections A–D below we will define $\mathcal{L}''$ on a fixed column $C(i) \in G_p \subset G$. The index i is fixed throughout these sections and will be suppressed. We will consider the labellings as maps from $\{j|\ 0 \leq j < H\}$ to $\{(k,l)|\ 1 \leq k \leq K, 0 \leq l < h(k)\}$.

A. Base and top floors: Let k_1, k_2 be such that $\mathcal{L}'(0) = (k_1, 0)$, and $\mathcal{L}'(H-1) = (k_2, h(k_2)-1)$. Let $\mathcal{L}''(j) = \mathcal{L}'(j)$ for $j \in [0, h(k_1))$ and $j \in [H - h(k_2), H)$.

B. Floors $h(k_1)$ through $(\overline{J}-1)$:

Let $\overline{J}$ be the smallest integer such that $\overline{J} > (J + h(k_1))$ and $\mathcal{L}(\overline{J}) = (k,0)$ for some k. Notice that $\overline{J} \leq (J + 2\max h(k)) < \delta H$.

There are coefficients $m_k \in \mathbb{N}$ such that $(\overline{J} - h(k_1)) = \sum m_k h(k)$. For each k, put the label $(k,0)$ on m_k of the floors in $[h(k_1), \overline{J})$, filling in labels on the remaining such that:

if $\mathcal{L}''(j) = (k,l)$ with $l < (h(k)-1)$ then $\mathcal{L}''(j+1) = (k, l+1)$,
if $\mathcal{L}''(j) = (k, h(k)-1)$ then $\mathcal{L}''(j+1) = (k',0)$ for some k'.

C. Floors $\overline{J}$ through $(\overline{L}-1)$:

We will define $\mathcal{L}''(j) = \mathcal{L}(j)$ for $j \in [\overline{J}, \overline{L})$. We will choose $\overline{L}$ to be as large as possible under the condition that with this definition, $(N_k''[0,\overline{L}) + N_k''[H-h(k_2), H))$ does not exceed $N_k'[0,H)$ for any k.

For all k and for $\overline{J} \leq L \leq H$, define

$$E_k(L) := N'_k[0,H) - (N''_k[0,\overline{J}) + N''_k[H - h(k_2), H) + N_k[\overline{J}, L)).$$

Notice that for all k, $E_k(L)$ is a non-increasing function of L. We will let $\overline{L}$ be the largest integer L such that $E_k(L) \geq 0$ for all k. (One can show that $E_k(\overline{J}) > 0$ for all k, and $E_k(H) < 0$ for some k, so $\overline{L}$ is well-defined. The proof that $E_k(\overline{J}) > 0$ for all k is contained in the following.)

CLAIM: $\overline{L} > (1 - \epsilon/2)H$.

Proof of Claim: We first estimate the summands of $E_k(\overline{L})$. Since our column is in G_p we have

$$N'_k[0,H) > (\beta_p(k) - \delta)H \tag{1}$$

for all k.

Also for all k, we have

$$N''_k[0,\overline{J}) + N''_k[H - h(k_2), H) \leq (J + 3\max h(k)) < \delta H. \tag{2}$$

Notice that if $\overline{J} \leq L < 2\delta H$ then for all k we have

$$\begin{aligned} E_k(L) > (\beta_p(k) - \delta)H - \delta H - N_k[\overline{J}, L) &> (\beta_p(k) - 2\delta)H - 2\delta H \\ &= (\beta_p(k) - 4\delta)H \\ &> \beta_p(k)(1 - 4\epsilon/6)H > 0. \end{aligned}$$

Therefore $\overline{L} \geq 2\delta H$, and again since our column is in G we have

$$N_k[\overline{J}, \overline{L}) \leq N_k[0, \overline{L}) < (\beta_p(k) + \delta)\overline{L} \tag{3}$$

for all k.

For some k, we have $E_k(\overline{L}) = 0$. Using (1), (2) and (3) we see that for this k,

$$0 = E_k(\overline{L}) > (\beta_p(k) - \delta)H - \delta H - (\beta_p(k) + \delta)\overline{L}.$$

Rearranging, we get

$$\begin{aligned} \overline{L} > [(\beta_p(k) - 2\delta)/(\beta_p(k) + \delta)]H &> (1 - 3\delta/\beta_p(k))H \\ &> (1 - \epsilon/2)H \end{aligned}$$

which ends the proof of Claim. ∎

D. Floors $\overline{L}$ through $(H - h(k_2))$:

We have $E_k(\overline{L}) = N'_k[0, H) - N''_k[0, \overline{L}) - N''_k[H - h(k_2), H)$ for all k. Multiplying both sides of this equation by $h(k)$, then summing over k, we see that $\sum_k E_k(\overline{L})h(k) = (H - \overline{L} - h(k_2))$. Therefore, there are exactly enough unlabelled floors left to label $E_k(\overline{L})$ of the floors in $[\overline{L}, H - h(k_2))$ with $(k, 0)$ for each k, and fill in labels on the remaining floors such that:

if $\mathcal{L}''(j) = (k, l)$ with $l < (h(k) - 1)$ then $\mathcal{L}''(j + 1) = (k, l + 1)$,

if $\mathcal{L}''(j) = (k, h(k) - 1)$ then $\mathcal{L}''(j + 1) = (k', 0)$ for some k'.

After labelling the remaining floors in this way we see that

$N''_k[0, H) = E_k(\overline{L}) + N''_k[0, \overline{L}) + N''_k[H - h(k), H) = N'_k[0, H)$, for all k.

We define $\mathcal{L}''$ in this manner for all $C(i) \in G$. We get $N''_k[0, H(i))_i = N'_k[0, H(i))_i$ for all i and k, and $\mathcal{L}'' = \mathcal{L}'$ on base$(\mathcal{T})$ and top$(\mathcal{T})$. All that remains is to show that $\mathcal{L}''$ and $\mathcal{L}$ differ on a set of measure less than ϵ with respect to all of the measures ν_p with $1 \leq p \leq m$. For $C(i) \in G$ we have $\mathcal{L} = \mathcal{L}''$ on $[\overline{J}(i), \overline{L}(i))$. We know that $\overline{J}(i) < \delta H(i) < (\epsilon/6)H(i)$ and $(H(i) - \overline{L}(i)) < (\epsilon/2)H(i)$. Therefore,

$$\begin{aligned}
\|\mathcal{L} - \mathcal{L}''\|_{\nu_p} &\leq \sum_{C(i)\in G} \left[\sum_{j=0}^{\overline{J}(i)-1} \nu_p(B(i,j)) + \sum_{j=\overline{L}(i)}^{H(i)-1} \nu_p(B(i,j)) \right] + \sum_{C(i)\notin G} \nu_p(C(i)) \\
&< \sum_{C(i)\in G} [(\overline{J}(i) + H(i) - \overline{L}(i))\nu_p(B(i,0))] + 2\delta \\
&< \sum_{C(i)\in G} [(\epsilon/6 + \epsilon/2)H(i)\nu_p(B(i,0))] + \epsilon/3 \\
&\leq (\epsilon/6 + \epsilon/2) + \epsilon/3 = \epsilon. \qquad \blacksquare
\end{aligned}$$

4.4.1. Selection of δ and Z. Let $0 < \epsilon < 1$ be given. From ϵ, we obtain a $0 < \delta < 1/2$ and an integer $Z_1(\delta)$ from the Labelling Lemma. Using this δ, we obtain $Z_2(\delta, S, \mu_p), Z_2(\delta, T, \nu_p)$ for each $1 \leq p \leq m$ from from the above lemma. Let Z be the maximum of $Z_1(\delta)$, $\max_p Z_2(\delta, S, \mu_p)$ and $\max_p Z_2(\delta, T, \nu_p)$. Then by the Labelling Lemma, we get the desired labelling $\mathcal{L}''$ and a multitower for T, τ'_1. It is by this process that the labelling $\mathcal{L}''$ is constructed, and hence a multitower τ'_1 for T such that $(\tau'_1, \mathcal{T}_2)$ is a nested pair, $(\tau'_1; \nu_p) \approx (\sigma; \mu_p)$ and $\|\tau'_1 - \tau\|_{\nu_p} < \epsilon/2$ for all $1 \leq p \leq m$.

4.5. Step 5 — Cleaning Up. Note that we are not quite done since it is not true that $(\Sigma_1; \mu_p) \approx (\mathcal{T}_2; \nu_p)$; instead, $\mathcal{T}_2$ is a refined copy of Σ_1.

Repairing this is a step that may look just technical, but, in fact, it plays a crucial role in the entire proof. The reader may have noticed that up to this moment we have never used the fact that the ergodic S-invariant measures μ_p, $p = 1, 2, \dots$ are distinct (hence, mutually singular). It is obvious, however, that without using this fact the proof cannot be done. Further, one may also note that until this point, the floors of $\mathcal{T}_2$ could have the property that they have positive measure with respect to only one measure.

Each base set $A(i, 0)$ in Σ_1 corresponds to a collection of base sets

$$B(k_i, 0), B(k_i + 1, 0), \dots, B(k_{i+1} - 1, 0)$$

in $\mathcal{T}_2$ where $\mu_p(A(i, 0)) = \nu_p(\bigcup_{j=k_i}^{k_{i+1}-1} B(j, 0))$. We will fix this, as in [18, p. 126], by partitioning each $A(i, 0)$ into clopen sets

$$A'(k_i, 0), A'(k_i + 1, 0), \dots, A'(k_{i+1} - 1, 0).$$

However (and this is the main difference between the situation in [18] and the situation here), we now need to control the values of all measures $\mu_1, \dots, \mu_m$ when we do this partitioning. Since the measures μ_p are mutually singular, the range of the vector measure $(\mu_1, \dots, \mu_m)$ is the entire m-dimensional cube. This allows us to choose the clopen sets $A'(k, 0)$ such that

$$\sum_k |\mu_p(A'(k, 0)) - \nu_p(B(k, 0))| H'(k) < \epsilon/2$$

for any $1 \le p \le m$. Let $A'(i, j) = S^j A'(i, 0)$ for $0 \le j < H'(i)$, and let $\Sigma = \langle A(i, j); H'(i); I' \rangle$. Now choose measurable sets $B'(i, 0) \subseteq \bigcup_{k=k_i}^{k_{i+1}-1} B(k, 0)$ such that

$$\nu_p(B'(i, 0)) = \mu_p(A'(i, 0)) \quad \text{and} \quad \sum_i |\nu_p(B'(i, 0) \bigtriangleup B(i, 0))| H'(i) < \epsilon/2.$$

Let $B'(i, j) = T^j B'(i, 0)$ for $0 \le j < H'(i)$ and $\mathcal{T} = \langle B'(i, j); H'(i); I' \rangle$. Finally, consider the labelling $\mathcal{L}''$ on $\mathcal{T}$, where $\mathcal{L}''(B'(i, j)) = \mathcal{L}''(B(i, j))$. This defines a new multitower τ' such that $(\tau'; \nu_p) \approx (\sigma; \mu_p)$ and $\|\tau' - \tau\|_{\nu_p} < \epsilon/2$ for $1 \le p \le m$. These are the final versions of the multitowers, and this completes the proof of Lemma 6 and, therefore, the proof of the main theorem.

5. Concluding remarks

The realization theorem for countable families of e.m.p.a.'s cannot be fully generalized to arbitrary (uncountable) families. The result of [7] shows that there are some descriptive conditions of the semicontinuity type that are necessary for topological, even nonminimal, realization of a family of e.m.p.a.'s, and the complete list of these conditions is unclear. What looks plausible is that the recent result of T. Downarowicz [4], which shows that the conditions for minimal realization and conditions for aperiodic realization are the same, can be strengthened "in the orbital direction". Namely, it may be true that every family that can be realized aperiodically can also be realized minimally *in the orbit equivalence class of any Cantor minimal system whose simplex of invariant measures is affinely homeomorphic to the simplex corresponding to the aperiodic realization.* This statement, if true, would give, in a sense, a common generalization of our results and the results of Downarowicz's paper [4].

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